There is a second way to multiply two vectors,

the cross product. The cross product of two vectors, $\vec{u}$ and $\vec{v}$, is denoted $\vec{u} \times \vec{v}$. There are many differences between the dot product and the cross product.

	Dot product	Cross product
Notation	$\vec{u} \cdot \vec{v}$	$\vec{u} \times \vec{v}$
Value	$\vec{u} \cdot \vec{v}$ is a scalar	$\vec{u} \times \vec{v}$ is a vector
Vanishes if	$\vec{u}$ and $\vec{v}$ are orthogonal	$\vec{u}$ and $\vec{v}$ are parallel
Works for	2D & 3D	3D only
Swaps & you get	$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$	$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

The cross product is somewhat a different beast. Let's go straight into the formula:

for $\vec{u}_1 = \langle x_1, y_1, z_1 \rangle$, $\vec{u}_2 = \langle x_2, y_2, z_2 \rangle$

$$\vec{u}_1 \times \vec{u}_2 = \langle y_1 z_2 - z_1 y_2,\ z_1 x_2 - x_1 z_2,\ x_1 y_2 - y_1 x_2 \rangle.$$

The formula is really tricky to memorize..

There are two ways to memorize.

1. xyzzy method

The x-component of $\vec{u_1} \times \vec{u_2}$ is

$$y_1 z_2 - z_1 y_2.$$

So x $\quad$ yz $\quad$ zy $\rightsquigarrow$ xyzzy.

The indices here are always 1 2 1 2.

The other components can be guessed by shifting variables. Namely, you permute the variables x, y, z in a cyclic order,

$$\begin{tikzcd} x \arrow[r] & y \arrow[dl] \\ z \arrow[u] & \end{tikzcd}$$

, and you get the y-component and the z-component.

$$\begin{array}{lcl}
x\text{-component} & = & y_1 z_2 - z_1 y_2 \\
\downarrow & & \downarrow\ \downarrow\quad\ \downarrow\ \downarrow \\
y\text{-component} & = & z_1 x_2 - x_1 z_2 \\
\downarrow & & \downarrow\ \downarrow\quad\ \downarrow\ \downarrow \\
z\text{-component} & = & x_1 y_2 - y_1 x_2.
\end{array}$$

2. Determinants (Makes sense only if you know matrices)

==Skip this== if you've never seen matrices before.

This is just a suggestion on how to memorize, and will not be tested.

$$\vec{u}_1 = \langle x_1, y_1, z_1 \rangle = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$$

$$\vec{u}_2 = \langle x_2, y_2, z_2 \rangle = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}, \text{ then}$$

$\vec{u}_1 \times \vec{u}_2$ is given by the determinant of the 3×3 matrix

$$\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}.$$

The determinant of a matrix can be computed as follows.

Ⓐ. Remove the first row, and consider 3 2×2 matrices formed by further removing each column:

$$\begin{vmatrix} \cancel{\vec{i}} & \cancel{\vec{j}} & \cancel{\vec{k}} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \rightsquigarrow \begin{vmatrix} \cancel{x_1} & y_1 & z_1 \\ \cancel{x_2} & y_2 & z_2 \end{vmatrix} \quad \begin{vmatrix} x_1 & \cancel{y_1} & z_1 \\ x_2 & \cancel{y_2} & z_2 \end{vmatrix} \quad \begin{vmatrix} x_1 & y_1 & \cancel{z_1} \\ x_2 & y_2 & \cancel{z_2} \end{vmatrix}$$

$$\rightsquigarrow \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \quad \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \quad \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

Ⓑ Multiply each term with the corresponding entry in the first row:

$$\rightsquigarrow \vec{i}\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \quad \vec{j}\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \quad \vec{k}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

Ⓒ Add all of them, with alternating signs + − + ..

$$\rightsquigarrow \quad +\vec{i}\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - \vec{j}\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + \vec{k}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

The determinant of a 2×2 matrix can be done via the same process.

Ⓐ. Remove the first row, and consider 2 1×1 matrices formed by further removing each column:

$$\begin{vmatrix} \cancel{y_1} & \cancel{z_1} \\ y_2 & z_2 \end{vmatrix} \rightsquigarrow \quad |\cancel{y_2}\ z_2| \qquad |y_2\ \cancel{z_2}|$$

Ⓑ Multiply each term with the corresponding entry in the first row:

$$\rightsquigarrow \quad y_1\,|z_2| \qquad z_1\,|y_2|$$

Ⓒ Add all of them, with alternating signs + − + ..

$$\rightsquigarrow +y_1z_2 - z_1y_2.$$

So we get

$$\vec{u}\times\vec{v} = (y_1z_2 - z_1y_2)\vec{i} - (x_1z_2 - z_1x_2)\vec{j} + (x_1y_2 - y_1x_2)\vec{k}.$$

THE END

Example For $\vec{u}=\langle 1,3,4\rangle$
$\vec{v}=\langle 2,7,-5\rangle$

(x=yz−zy!)

$$\vec{u}\times\vec{v}=\langle 3(-5)-4(7),\ 4\cdot 2-1\cdot(-5),\ 1\cdot 7-3\cdot 2\rangle$$

$$=\langle -43,\ 13,\ 1\rangle.$$

The cross product has some weird properties!

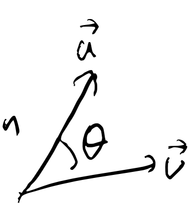
0. Obviously $\vec{0}\times\vec{u}$ and $\vec{u}\times\vec{0}$ are both zero vector, $\vec{0}$.
1. $\vec{u}\times\vec{v}=-\vec{v}\times\vec{u}$.
2. $\vec{u}\times\vec{v}=0$ if $\vec{u}$ and $\vec{v}$ are parallel.
   Conversely, $\vec{u}\times\vec{v}=0$ implies that $\vec{u}$ and $\vec{v}$ are parallel.
3. $\vec{u}\times\vec{v}$ as a vector is orthogonal to both $\vec{u}$ and $\vec{v}$.
   Namely, $\vec{u}\cdot(\vec{u}\times\vec{v})=0$ and $\vec{v}\cdot(\vec{u}\times\vec{v})=0$.

Example In the above example,

$$\vec{u}\cdot(\vec{u}\times\vec{v})=1\cdot(-43)+3\cdot 13+4\cdot 1=-43+39+4=0$$

$$\vec{v}\cdot(\vec{u}\times\vec{v})=2\cdot(-43)+7\cdot 13+(-5)\cdot 1=-86+91-5=0.$$

4. If $\vec{u}$ and $\vec{v}$ forms an angle of θ, then



$$|\vec{u}\times\vec{v}|=|\vec{u}|\cdot|\vec{v}|\sin\theta.$$

Remember that $|\vec{u}\cdot\vec{v}| = |\vec{u}|\cdot|\vec{v}|\cos\theta$.

In the case of dot product, $\vec{u}\cdot\vec{v}=0$

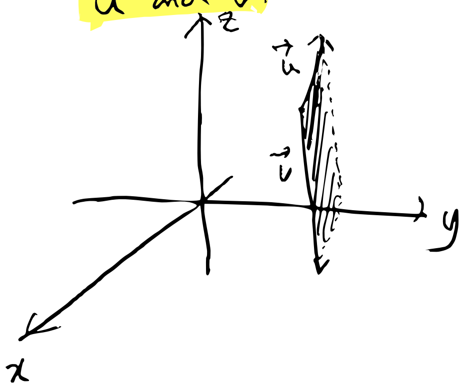
if $\cos\theta = 0$, or if $\theta = \frac{\pi}{2}$ $(90°)$, namely when they're orthogonal

In the case of cross product, $\vec{u}\times\vec{v}=0$

if $\sin\theta = 0$, or if $\theta = 0$ or π $(180°)$, namely when they're parallel.

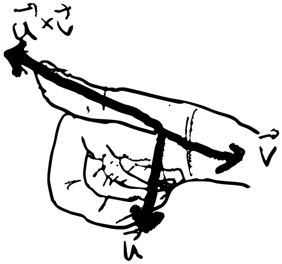
5. From the above formula,

$|\vec{u}\times\vec{v}|$ is the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$.



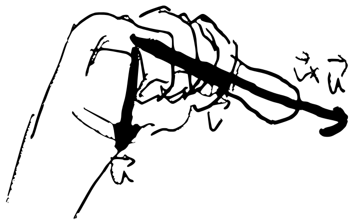
So, the area of the triangle formed by $\vec{u}$ and $\vec{v}$ is $\frac{1}{2}|\vec{u}\times\vec{v}|$, being the half of the area of the parallelogram.

6. The direction of $\vec{u}\times\vec{v}$ follows the right hand rule: among the two directions orthogonal to both $\vec{u}$ and $\vec{v}$, it's the direction of the thumb as your fingers curl from $\vec{u}$ to $\vec{v}$.



(sorry about the picture)

This is another illustration of why $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$.



(sorry about the picture)

<u>Example</u> $\vec{i}\times\vec{j}$ has length $|\vec{i}|\cdot|\vec{j}|\cdot\sin\frac{\pi}{2} = 1$,

and has the direction the same as the positive z-direction,

so $\vec{k} = \vec{i}\times\vec{j}$. Indeed $\langle 1,0,0\rangle \times \langle 0,1,0\rangle$

$= \langle 0\cdot 0 - 0\cdot 1, 0\cdot 0 - 1\cdot 0, 1\cdot 1 - 0\cdot 0\rangle = \langle 0,0,1\rangle$.

Warning It is extremely important to not mix up vectors with scalars.

For example, you can't add a scalar to a vector.

Also, you can't divide by vectors.

Exercise Suppose $\vec{u}, \vec{v}, \vec{w}$ are vectors, and a, b are scalars. Check if the following expressions are valid and explain why.

1. $a(\vec{u} \times \vec{v}) \times \vec{w}$
2. $(\vec{u} + |\vec{v}|) \cdot b\vec{w}$
3. $(a+b)a\vec{u} + \vec{v} \cdot \vec{w}$
4. $\dfrac{|\vec{v}|}{\vec{w}}(\vec{u} \cdot \vec{v})$
5. $\dfrac{(\vec{u} \cdot \vec{v})^2}{a|\vec{w}|}\vec{u} + b(\vec{v} \times \vec{w}) \times \vec{u}$

A fun fact is that, just as the cross product computes the area of a parallelogram, the cross product and the dot product combined can compute the volume of a parallelepiped (3D version of parallelogram).

$|\vec{u}\cdot(\vec{v}\times\vec{w})|$ computes the paralelepiped formed by the three vectors $\vec{u}, \vec{v}, \vec{w}$.